# **Lattice-based sum of t-norms on bounded lattices**

Abstract

The concept of [ordinal sums](https://www.sciencedirect.com/topics/mathematics/ordinal-sum) in the sense of Clifford have long been blamed for their limitations in constructing new t-norms including inability to cope with general bounded [lattices](https://www.sciencedirect.com/topics/mathematics/lattices). Motivated by this observation, and based on the lattice-based sum of [lattices](https://www.sciencedirect.com/topics/mathematics/lattices) that has been recently introduced by El-Zekey et al., we propose a new sum-type construction of t-norms, called a *lattice-based sum of t-norms*, for building new t-norms on bounded lattices from given ones. The proposed sum is generalizing the well-known ordinal and horizontal sum constructions of t-norms by allowing for lattice ordered index sets. We demonstrate that, like the ordinal sum of t-norms, the lattice-based sum of t-norms can be generalized using as summands so-called t-subnorms, still leading to a t-norm. Subsequently, we apply the results for constructing several new families of t-norms and t-subnorms on bounded lattices. In the same spirit, by the duality, we will also introduce *lattice-based sums of t-conorms* and *t-subconorms*.

Introduction

Triangular norms (t-norms for short) on the unit interval were originally studied by Schweizer and Sklar (see e.g. [35]) in the framework of probabilistic metric spaces. Later on, they turned out to be interpretations of the conjunction in fuzzy logic and, subsequently, for the intersection of fuzzy sets. They have also proven to be useful in many other fields like decision making, statistics as well as in the theories of non-additive measures and cooperative games. For a comprehensive treatment on t-norms on the unite interval we refer to the monographs [4], [24].

One of typical constructions for t-norms on the unit interval is the ordinal sum in the sense of Clifford [5]. It provided a method to construct a new t-norm from a given system of t-norms indexed by a linearly ordered index set. As shown in [18], this ordinal sum construction can be generalized using as summands so-called t-subnorms, still leading to a t-norm. It is worth mentioning that ordinal sums are a construction method but also a representation tool in the framework of associative conjunctions in many-valued logics and intersections in fuzzy set theory [8], [13], [18], [24], [25], [26], [34].

Due to the close connection between order theory and fuzzy set theory, the unit interval was replaced by some more general structure, for example, a bounded lattice (see e.g. [12]). Therefore, recently an increasing interest of t-norms on bounded lattices can be observed, see e.g. [3], [7], [10], [11], [16], [17], [19], [23], [24], [29], [30], [32], [33], [37] and many others.

Considering bounded lattices, the concept of ordinal sums in the sense of Clifford [5] have long been blamed for their limitations in constructing new t-norms including inability to cope with general bounded lattices. On the one hand, as observed in [32], in general, an ordinal sum like construction of t-norms may not work on bounded lattices. In [32] (see also [29], [33]) constraints on bounded lattices guaranteeing that the ordinal sum operation yields again a t-norm on the lattices have been revealed. On the other hand, it is known that the ordinal sum of t-norms in the sense of Clifford [5] whose carriers are bounded lattices is again a t-norm with a carrier equal to the ordinal sum (in the sense of Birkhoff [1]) of the summand lattices. However, due to the ordinality (i.e. the linear order of the index set involved), there exist ordinal sum like construction of t-norms yielding again a t-norm on bounded lattices which are not an ordinal sum of some of their sublattices, see [32]. In summary, there is a need for a new sum-type construction generalizing the ordinal sum construction of t-norms and cope very well with general bounded lattices.

Motivated by these observations, in this contribution we propose a new sum-type construction of t-norms for building new t-norms on bounded lattices from given ones. We will call such a proposed sum-type construction a *lattice-based sum of t-norms*. The proposed sum is based on *the lattice-based sum of lattices* that has been recently introduced by El-Zekey et al. (see [9]), for building new (bounded) lattice ordered sets from fixed ones. The lattice-based sum construction of t-norms is generalizing the well-known ordinal sum constructions of t-norms by allowing for lattice ordered index sets instead of linearly ordered index set. It extends also the *horizontal sum* based on an unstructured index set (i.e., any two distinct indices are incomparable), see [9]. We demonstrate that, like the ordinal sum of t-norms (see [18]), the lattice-based sum of t-norms can be generalized using as summands so-called t-subnorms, still leading to a t-norm. Subsequently, we apply the results for constructing several new families of t-norms and t-subnorms on bounded lattices. In the same spirit, by the duality, we will also introduce *lattice-based sums of t-conorms* and *t-subconorms* showing that all results given for t-norms (t-subnorms) remain valid for t-conorms (t-subconorms) with the obvious changes where necessary.

This paper is organized as follows. The next section is a preliminary that includes the basic concepts and definitions required in this paper. In Section 3 we briefly recall the lattice-based sum construction of lattice ordered sets. In Section 4, we introduce the lattice-based sum of t-norms and t-subnorms. In Section 5, we apply the latticed-based sum theorems, from Section 4, for constructing several new t-norms and t-subnorms on bounded lattices. The lattice-based sums of t-conorms and t-subconorms are presented in Section 6. We close this contribution by a short summary and further perspectives.

Section snippets

Preliminaries

Recall that [1], [6] a *lattice-ordered set* is a *partially ordered set* (�,⪯�) in which each two-element subset {�,�} has an infimum, denoted �∧�, and a supremum, denoted �∨�. A bounded lattice (�,⪯�) is a lattice-ordered set which has the top and bottom elements which are written as ⊤� and ⊥�, respectively, that is, ⊥�⪯��⪯�⊤�, for all *x* ∈ *L*. If �,�∈� such that �⪯�� but �≠�, then we will write �≺��.

**Definition 1**

see [7], [24]

Let (�,⪯�,⊥�,⊤�) be a bounded lattice. A binary operation �:�2⟶� is called a *triangular norm*

Lattice-based sum of bounded lattices

In this section we recall some definitions and results that will be necessary for our work. Some details will be omitted, but an appropriate description of them can be found in [9]

In the sequel, (Λ,⊑) denotes a *(bounded) lattice-ordered index set*. The top and bottom elements of Λ will be denoted by ⊤Λ and ⊥Λ, respectively. (��,∧�,∨�) denotes a *bounded lattice-ordered set* for some �∈Λ, where ∧� and ∨� are the meet and join operations on ��, respectively. The top and bottom elements of �� will be

Lattice-based sum of t-norms on bounded lattices

**Lemma 4**

*Consider a lattice-ordered index set* (�,⊑) *and a lattice-based sum of bounded lattices* (�,∧,∨)=⨁�∈�(��,∧�,∨�)*. Assume that there exists* �1,�2∈� *such that there is no* �∈� *such that* {�1,�2}⊆��*. Then*

* *(i)*

*If* �1≺��2*, then there exists* �1,�2∈� *such that* (�1,�2)∈(��1,��2) *with* �1⊏�2*, and for all* (�1,�2)∈(��1,��2) *we have* �1⪯��2*.*

* *(ii)*

*If* �1∥�2*, then for all* �1∈��1 *and* �2∈��2*,* �1∥�2*, furthermore for all* �1∈��1\{⊤�1,⊥�1} *and* �2∈��2\{⊤�2,⊥�2} *we have* �1∥�2*.*

* *(iii)*

*If* �1∥�2*, then necessarily* �1∧�2∈{⊥�,⊤�} *for some* �∈�*.*

**Proof**

Lattice-based sum construction of t-norms on bounded lattices

The aim of this section is to apply the latticed-based sum theorems, from Section 4, for constructing several new t-norms and t-subnorms on bounded lattices. Perhaps, the most important consequence is that a new construction method for t-norms on bounded lattices arise.

**Proposition 7**

*Consider a bounded lattice-ordered index set* (�,⊑) *and a lattice-based sum of bounded lattices* (�,∧,∨)=⨁�∈�(��,∧�,∨�)*. Fix an* ��∈�� *for each* �∈�*. Then the following operation is a t-norm on L.*�(�,�)={�∧�∧��,if(�,�)∈(��\{⊤�})2�∧�,

Lattice-based sum of t-conorms on bounded lattices

In the same spirit as in Definition 7 we will also introduce in this section the *lattice-based sum of t-subconorms* and then the *lattice-based sum of t-conorms*.

Recall that, from an axiomatical point of view, t-norms and t-conorms differ only with respect to their respective boundary conditions, i.e. t-norms have the top element as unit, while t-conorms have the bottom element as unit. Analogously, t-subnorms and t-subconorms differ only with respect to their respective range conditions, i.e.

Conclusions and future work

In this contribution, we submit to the reader a proposal of a new construction method for t-norms which is promising for the further development of the theory of t-norms on bounded lattices, generalizing both ordinal and horizontal sums of t-norms. We generalized the well-known ordinal sum technique of t-norms to what we call lattice-based sum of t-norms by allowing for lattice ordered index set instead of linearly ordered index set, showing that the lattice-based sum of t-norms is again a